

ENERGY EVOLUTION OF MULTI-SYMPLECTIC METHODS FOR MAXWELL EQUATIONS WITH PERFECTLY MATCHED LAYER BOUNDARY^{a)}

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In this paper, we consider the energy evolution of multi-symplectic methods for three-dimensional (3D) Maxwell equations with perfectly matched layer boundary, and present the energy evolution laws of Maxwell equations under the discretization of multi-symplectic Yee method and general multi-symplectic Runge-Kutta methods.

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I. INTRODUCTION

Maxwell equations are the most foundational equations in electromagnetism and are widely applied to many application fields, such as aeronautics, electronics, and biology^{12,14}, etc. They are mathematical expressions of the natural laws correlative fields, such as Ampère's law and Faraday's law¹³. On the other hand, in lossless medium, the electromagnetic energy of the wave is constant at different times⁴. As we all know, to preserve the energy is greatly important in constructing numerical schemes for different physical problems. However, numerical methods, with some boundary conditions, can not preserve the energy exactly in general cases. Therefore, it is important and necessary to investigate the energy evolution of Maxwell equations under numerical discretization with some boundary conditions. The purpose of this paper is to study the energy evolution of multi-symplectic methods for 3D Maxwell's equations with perfectly matched layer (PML) boundary.

It has been recognized that symplectic structure-preserving numerical methods have significant superiority than non-symplectic methods in numerical solving Hamiltonian ordinary differential equations (ODEs) and Hamiltonian partial differential equations (PDEs)³. At the end of last century, symplectic integrators have been generalized to multi-symplectic ones^{4-9,11}. And multi-symplectic integrators have been applied to Maxwell equations. For examples,¹¹ discussed the self-adjointness of the Maxwell equations with variable coefficients ϵ and μ , and showed that the equations have the multi-symplectic structure.⁴ firstly proposed an unconditionally stable, energy-conserved, and computational efficiently scheme for two-dimensional (2D) Maxwell equations with an isotropic and lossless medium. The further analysis in

the case of 3D was studied in⁵. Meanwhile,⁹ proposed a kind of splitting multi-symplectic integrators method for Maxwell equations in three dimensions, which was proved to be unconditionally stable, non-dissipative, and of first order accuracy in time and second order accuracy in space.

It is well known that the PML boundary conditions are widely applied to the numerical simulation Maxwell equations. In 1993, Berenger^{1,2} firstly proposed the PML technique, which is based on modifying the PDEs away from all physical boundaries such that absorbing outgoing waves from the computation domain. It is a simple and straightforward technique, easily implemented for both two and three space dimensions using either cartesian or cylindrical coordinates. However, to the best of our knowledge, the investigation of multi-symplectic methods for Maxwell equations with PML boundary does not exist. In this paper, inspired by this problem, we investigate the energy evolution of general multi-symplectic methods for Maxwell equations with Berenger's PML boundary.

The rest of this paper is organized as follows. In Section 2, we begin with some preliminary results about 3D Maxwell equations and Berenger's PML systems. An equivalent formulation to Berenger's PML systems is introduced in Section 3. In Section 4, we present the energy evolution laws of multi-symplectic Yee method and general multi-symplectic Runge-Kutta methods for 3D Maxwell equations with PML boundary.

II. PRELIMINARY RESULTS

Notations. We denote by (\cdot, \cdot) the L^2 scalar product, $\|\cdot\|_{H^s}$ the norm in H^s .

A. 3D Maxwell equations

For a linear homogeneous medium within linear isotropic material with the permittivity ϵ and the permeability μ , the

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scattering of electromagnetic waves without the charges or the currents can be described by the 3D Maxwell equations in curl formulation

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H}, \quad (1)$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E}, \quad (2)$$

where $\mathbf{E} = (E_x, E_y, E_z)^T$ and $\mathbf{H} = (H_x, H_y, H_z)^T$ represent the electric field and the magnetic field, respectively. The domain $\Omega \times [0, T] = [0, a] \times [0, b] \times [0, c] \times [0, T]$ under consideration is occupied by this medium and surrounded by perfect conductors.

The curl equations (1) and (2) can be written as the componentwise formula

$$\frac{\partial}{\partial t} \begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \\ \frac{1}{\varepsilon} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \\ \frac{1}{\varepsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \\ \frac{1}{\mu} \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \\ \frac{1}{\mu} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\ \frac{1}{\mu} \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \end{bmatrix}. \quad (3)$$

When the medium is lossless, then by Green's formula it gets the following invariants:

$$\text{Energy I: } \int_{\Omega} (\varepsilon |\mathbf{E}(\mathbf{x}, t)|^2 + \mu |\mathbf{H}(\mathbf{x}, t)|^2) d\Omega = \text{Constant},$$

$$\text{Energy II: } \int_{\Omega} (\varepsilon \left| \frac{\partial \mathbf{E}(\mathbf{x}, t)}{\partial t} \right|^2 + \mu \left| \frac{\partial \mathbf{H}(\mathbf{x}, t)}{\partial t} \right|^2) d\Omega = \text{Constant}.$$

The first invariant is called Poynting theorem in electromagnetism and it can be easily verified, and the second is a little more complex. For more details, see⁴.

In the 2D transverse electric (TE) polarization case, the electric and magnetic field read $\mathbf{E} = (E_x, E_y, 0)^T$, $\mathbf{H} = (0, 0, H_z)^T$. Therefore, the Maxwell equations (1) and (2) become

$$\begin{cases} \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, \\ \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}, \\ \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}. \end{cases} \quad (4)$$

Let $Z = (H_x, H_y, H_z, E_x, E_y, E_z)^T$. Then the componentwise formula (3) is multi-symplectic, i.e.,

$$MZ_t + K_1 Z_x + K_2 Z_y + K_3 Z_z = 0, \quad (5)$$

where

$$M = \begin{pmatrix} 0 & -I_{3 \times 3} \\ I_{3 \times 3} & 0 \end{pmatrix}, K_p = \begin{pmatrix} \varepsilon^{-1} R_p & 0 \\ 0 & \mu^{-1} R_p \end{pmatrix}, \quad \forall p = 1, 2, 3.$$

The sub-matrix $I_{3 \times 3}$ is a 3×3 identity matrix and

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The multi-symplectic formulation in (5) preserves the following multi-symplectic structure

$$\frac{\partial}{\partial t} dZ \wedge M dZ + \sum_{p=1}^3 \frac{\partial}{\partial x_p} dZ \wedge K_p dZ = 0, \quad (6)$$

where dZ is the solution of the variational equation associated with (5).

Let $[\partial_t]_{i,j,k}^n$ and $[\partial_{x_p}]_{i,j,k}^n$ denote the discretization of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_p}$ (for $p = 1, 2, 3$), where n is the temporal index and i, j, k are the spatial indices in the discrete system. For the multi-symplectic Hamiltonian PDEs (5), we consider the following numerical discretization

$$M[\partial_t]_{i,j,k}^n Z_{i,j,k}^n + \sum_{p=1}^3 K_p [\partial_{x_p}]_{i,j,k}^n Z_{i,j,k}^n = 0. \quad (7)$$

Definition 1 The numerical method (7) is multi-symplectic if it preserves the discrete version of the multi-symplectic structure in (6). That is,

$$[\partial_t]_{i,j,k}^n \left(dZ_{i,j,k}^n \wedge M dZ_{i,j,k}^n \right) + \sum_{p=1}^3 [\partial_{x_p}]_{i,j,k}^n \left(dZ_{i,j,k}^n \wedge K_p dZ_{i,j,k}^n \right) = 0. \quad (8)$$

From the multi-symplectic Hamiltonian formulation given by (5), of which its solution preserves the multi-symplectic structure (6). Now, we list several multi-symplectic numerical schemes applied to Maxwell equations given in this section.

• Yee method

This method is the basis of the highly popular numerical methods known as the FDTD methods¹³. Yee method is constructed by central difference in both space and time based on a half-step staggered grid. It is a second-order method and is conditionally stable. Recently,¹⁰ showed that Yee method is multi-symplectic by the discrete variational principle, so we call it the multi-symplectic Yee method.

• Multi-symplectic Runge-Kutta methods

Applying the symplectic Runge-Kutta methods in both time and space to Maxwell equations (5) leads to the multi-symplectic Runge-Kutta methods.⁷ presented the sufficient conditions of multi-symplecticity Runge-Kutta methods for Hamiltonian PDEs.

B. Berenger's PML system for 3D Maxwell equations

In the PML medium, each component of electromagnetic field is split into two parts. In cartesian coordinates, the six components yield 12 subcomponents denoted by $E_{xy}, E_{xz}, E_{yx},$

$E_{yz}, E_{zx}, E_{zy}, H_{xy}, H_{xz}, H_{yx}, H_{yz}, H_{zx}, H_{zy}$, and the Maxwell equations read,

$$\varepsilon \frac{\partial E_{xy}}{\partial t} + \sigma_y E_{xy} = \frac{\partial (H_{zx} + H_{zy})}{\partial y}, \quad (9)$$

$$\varepsilon \frac{\partial E_{xz}}{\partial t} + \sigma_z E_{xz} = -\frac{\partial (H_{yz} + H_{yx})}{\partial z}, \quad (10)$$

$$\varepsilon \frac{\partial E_{yz}}{\partial t} + \sigma_z E_{yz} = \frac{\partial (H_{xy} + H_{xz})}{\partial z}, \quad (11)$$

$$\varepsilon \frac{\partial E_{yx}}{\partial t} + \sigma_x E_{yx} = -\frac{\partial (H_{zx} + H_{zy})}{\partial x}, \quad (12)$$

$$\varepsilon \frac{\partial E_{zx}}{\partial t} + \sigma_x E_{zx} = \frac{\partial (H_{yz} + H_{yx})}{\partial x}, \quad (13)$$

$$\varepsilon \frac{\partial E_{zy}}{\partial t} + \sigma_y E_{zy} = -\frac{\partial (H_{xy} + H_{xz})}{\partial y}, \quad (14)$$

$$\mu \frac{\partial H_{xy}}{\partial t} + \sigma_y^* H_{xy} = -\frac{\partial (E_{zx} + E_{zy})}{\partial y}, \quad (15)$$

$$\mu \frac{\partial H_{xz}}{\partial t} + \sigma_z^* H_{xz} = \frac{\partial (E_{yz} + E_{yx})}{\partial z}, \quad (16)$$

$$\mu \frac{\partial H_{yz}}{\partial t} + \sigma_z^* H_{yz} = -\frac{\partial (E_{xy} + E_{xz})}{\partial z}, \quad (17)$$

$$\mu \frac{\partial H_{yx}}{\partial t} + \sigma_x^* H_{yx} = \frac{\partial (E_{zx} + E_{zy})}{\partial x}, \quad (18)$$

$$\mu \frac{\partial H_{zx}}{\partial t} + \sigma_x^* H_{zx} = -\frac{\partial (E_{yz} + E_{yx})}{\partial x}, \quad (19)$$

$$\mu \frac{\partial H_{zy}}{\partial t} + \sigma_y^* H_{zy} = \frac{\partial (E_{xy} + E_{xz})}{\partial y}, \quad (20)$$

where the parameters $(\sigma_x, \sigma_y, \sigma_z, \sigma_x^*, \sigma_y^*, \sigma_z^*)$ are homogeneous to electric and magnetic conductivities.

If $\sigma_x = \sigma_y = \sigma_z$ and $\sigma_x^* = \sigma_y^* = \sigma_z^* = 0$, then (9)-(20) yield the classical Maxwell equations (1)-(2). Thus, the absorbing medium defined by (9)-(20) holds as particular cases of all usual media (vacuum, conductive media).

The 3D PML technique is a straightforward generalization of the 2D case¹. The Maxwell equations are solved by the FDTD method within a computational domain surrounded by an absorbing layer which is an aggregate of PML media.

III. ONE FORMULATION EQUIVALENT TO BERENGER'S FORMULATION

Berenger's formulation involves a splitting of the unknown electromagnetic fields. The idea of¹⁵ is to restore the usual operator by introducing a new variable.

Let us consider the Berenger's PML parrel xoy -plane, i.e., $\sigma_x = \sigma_x^* = 0$ and $\sigma_y = \sigma_y^* = 0$. For simplicity, we assume that $\varepsilon = \mu \equiv 1$ and $\sigma_z = \sigma_z^* \equiv \sigma$, where σ is a constant that does not depend on x, y, z, t . Then Berenger's systems (9)-(20) of

the 3D Maxwell equations can be rewritten as

$$\frac{\partial E_{xy}}{\partial t} = \frac{\partial (H_{zx} + H_{zy})}{\partial y}, \quad (21)$$

$$\frac{\partial E_{xz}}{\partial t} + \sigma E_{xz} = -\frac{\partial (H_{yz} + H_{yx})}{\partial z}, \quad (22)$$

$$\frac{\partial E_{yz}}{\partial t} + \sigma E_{yz} = \frac{\partial (H_{xy} + H_{xz})}{\partial z}, \quad (23)$$

$$\frac{\partial E_{yx}}{\partial t} = -\frac{\partial (H_{zx} + H_{zy})}{\partial x}, \quad (24)$$

$$\frac{\partial E_{zx}}{\partial t} = \frac{\partial (H_{yz} + H_{yx})}{\partial x}, \quad (25)$$

$$\frac{\partial E_{zy}}{\partial t} = -\frac{\partial (H_{xy} + H_{xz})}{\partial y}, \quad (26)$$

$$\frac{\partial H_{xy}}{\partial t} = -\frac{\partial (E_{zx} + E_{zy})}{\partial y}, \quad (27)$$

$$\frac{\partial H_{xz}}{\partial t} + \sigma H_{xz} = \frac{\partial (E_{yz} + E_{yx})}{\partial z}, \quad (28)$$

$$\frac{\partial H_{yz}}{\partial t} + \sigma H_{yz} = -\frac{\partial (E_{xy} + E_{xz})}{\partial z}, \quad (29)$$

$$\frac{\partial H_{yx}}{\partial t} = \frac{\partial (E_{zx} + E_{zy})}{\partial x}, \quad (30)$$

$$\frac{\partial H_{zx}}{\partial t} = -\frac{\partial (E_{yz} + E_{yx})}{\partial x}, \quad (31)$$

$$\frac{\partial H_{zy}}{\partial t} = \frac{\partial (E_{xy} + E_{xz})}{\partial y}. \quad (32)$$

Let $E_{xy}, E_{xz}, E_{yx}, E_{yz}, E_{zx}, E_{zy}, H_{xy}, H_{xz}, H_{yx}, H_{yz}, H_{zx}, H_{zy}$ be a solution of Berenger's system (21)-(32) with initial conditions $\mathbf{E}^0, \mathbf{H}^0$. Adding (25) and (26), (31) and (32), respectively, we can obtain that

$$\frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \quad (33)$$

$$\frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}. \quad (34)$$

Applying $\partial_t + \sigma$ to (21), ∂_t to (22), and adding the two terms give

$$\partial_t (\partial_t + \sigma) E_{xy} + \partial_t (\partial_t + \sigma) E_{xz} + \partial_t \partial_z H_y - (\partial_t + \sigma) \partial_y H_z = 0.$$

Since σ does not depend on y , the operators $\partial_t + \sigma$ and ∂_y commute, we get by setting $E_x = E_{xy} + E_{xz}$

$$\partial_t [(\partial_t + \sigma) E_x + \partial_z H_y] - \partial_y (\partial_t + \sigma) H_z = 0.$$

In order to transform the last term into a time derivative, we introduce a new variable \widetilde{H}_z satisfying

$$(\partial_t + \sigma) H_z = \partial_t \widetilde{H}_z, \quad (35)$$

then

$$\partial_t [(\partial_t + \sigma) E_x + \partial_z H_y - \partial_y \widetilde{H}_z] = 0.$$

If we make the assumption that at $t = 0$,

$$(\partial_t E_x)^0 + \sigma E_x^0 + \partial_z H_y^0 - \partial_y \widetilde{H}_z^0 = 0, \quad (36)$$

it follows

$$(\partial_t + \sigma)E_x + \partial_z H_y - \partial_y \widetilde{H}_z = 0. \quad (37)$$

Note that (35) can not completely determine \widetilde{H}_z . We have to prescribe an initial value of \widetilde{H}_z , say

$$\widetilde{H}_z^0 = H_z^0, \quad (38)$$

which in particular implies that $\widetilde{H}_z \equiv H_z$ if $\sigma = 0$.

A similar process, it implies

$$(\partial_t + \sigma)E_y + \partial_x H_z - \partial_z \widetilde{H}_x = 0. \quad (39)$$

Similarly, it follows from introducing a new variable \widetilde{E}_z satisfying

$$(\partial_t + \sigma)E_z = \partial_t \widetilde{E}_z, \quad (40)$$

that

$$(\partial_t + \sigma)H_x - \partial_z E_y + \partial_y \widetilde{E}_z = 0, \quad (41)$$

$$(\partial_t + \sigma)H_y + \partial_z E_x - \partial_x \widetilde{E}_z = 0. \quad (42)$$

In order to make the calculations of the next sections more readable, it is useful to adopt new notations for E_z and \widetilde{E}_z , H_z and \widetilde{H}_z .

- E_z is denoted by E_z^* , \widetilde{E}_z is denoted by E_z ;

- H_z is denoted by H_z^* , \widetilde{H}_z is denoted by H_z .

Then the un-splitting formulation (21)-(32) can be rewritten as

$$\begin{cases} (\partial_t + \sigma)E_x + \partial_z H_y - \partial_y H_z = 0, & (a) \\ (\partial_t + \sigma)E_y + \partial_x H_z - \partial_z H_x = 0, & (b) \\ \partial_t E_z^* + \partial_y H_x - \partial_x H_y = 0, & (c) \\ (\partial_t + \sigma)E_z^* = \partial_t E_z, & (d) \\ (\partial_t + \sigma)H_x + \partial_y E_z - \partial_z E_y = 0, & (e) \\ (\partial_t + \sigma)H_y + \partial_z E_x - \partial_x E_z = 0, & (f) \\ \partial_t H_z^* + \partial_x E_y - \partial_y E_x = 0, & (g) \\ (\partial_t + \sigma)H_z^* = \partial_t H_z. & (h) \end{cases} \quad (43)$$

IV. ENERGY EVOLUTION OF THE MULTI-SYMPLECTIC METHODS FOR 3D MAXWELL EQUATIONS

In this section, we apply two multi-symplectic methods to discrete 3D Maxwell equations.

1) Multi-symplectic Yee method.

We introduce the difference operators ($k = n$, or $k = n + \frac{1}{2}$)

$$\begin{aligned} (D_{\Delta t} U)^k &= \frac{U^{k+\frac{1}{2}} - U^{k-\frac{1}{2}}}{\Delta t}, & (D_{\Delta x} U)_k &= \frac{U_{k+\frac{1}{2}} - U_{k-\frac{1}{2}}}{\Delta x}, \\ (D_{\Delta y} U)_k &= \frac{U_{k+\frac{1}{2}} - U_{k-\frac{1}{2}}}{\Delta y}, & (D_{\Delta z} U)_k &= \frac{U_{k+\frac{1}{2}} - U_{k-\frac{1}{2}}}{\Delta z}, \\ \overline{U}^k &= \frac{U^{k+\frac{1}{2}} + U^{k-\frac{1}{2}}}{2}, \\ (D_{\Delta t}^\sigma U)_{\alpha,\beta,\gamma}^n &= \frac{U_{\alpha,\beta,\gamma}^{n+\frac{1}{2}} - U_{\alpha,\beta,\gamma}^{n-\frac{1}{2}}}{\Delta t} + \sigma \frac{U_{\alpha,\beta,\gamma}^{n+\frac{1}{2}} + U_{\alpha,\beta,\gamma}^{n-\frac{1}{2}}}{2} \\ &= (D_{\Delta t} U)_{\alpha,\beta,\gamma}^n + \sigma \overline{U}_{\alpha,\beta,\gamma}^n. \end{aligned} \quad (44)$$

With these notations, multi-symplectic Yee method for (43) is

$$(D_{\Delta t}^\sigma E_x)_{i+1/2,j,k}^{n+1/2} + (D_{\Delta x} H_y)_{i+1/2,j,k}^{n+1/2} = (D_{\Delta y} H_z)_{i+1/2,j,k}^{n+1/2}, \quad (45)$$

$$(D_{\Delta t}^\sigma E_y)_{i,j+1/2,k}^{n+1/2} + (D_{\Delta x} H_z)_{i,j+1/2,k}^{n+1/2} = (D_{\Delta z} H_x)_{i,j+1/2,k}^{n+1/2}, \quad (46)$$

$$(D_{\Delta t} E_z^*)_{i,j,k+1/2}^{n+1/2} + (D_{\Delta y} H_x)_{i,j,k+1/2}^{n+1/2} = (D_{\Delta x} H_y)_{i,j,k+1/2}^{n+1/2}, \quad (47)$$

$$(D_{\Delta t}^\sigma E_z^*)_{i,j,k+1/2}^{n+1/2} = (D_{\Delta t} E_z)_{i,j,k+1/2}^{n+1/2}, \quad (48)$$

$$(D_{\Delta t}^\sigma H_x)_{i,j+1/2,k+1/2}^n + (D_{\Delta y} E_z)_{i,j+1/2,k+1/2}^n = (D_{\Delta z} E_y)_{i,j+1/2,k+1/2}^n, \quad (49)$$

$$(D_{\Delta t}^\sigma H_y)_{i+1/2,j,k+1/2}^n + (D_{\Delta z} E_x)_{i+1/2,j,k+1/2}^n = (D_{\Delta x} E_z)_{i+1/2,j,k+1/2}^n, \quad (50)$$

$$(D_{\Delta t} H_z^*)_{i+1/2,j+1/2,k}^n + (D_{\Delta x} E_y)_{i+1/2,j+1/2,k}^n = (D_{\Delta y} E_x)_{i+1/2,j+1/2,k}^n, \quad (51)$$

$$(D_{\Delta t}^\sigma H_z^*)_{i+1/2,j+1/2,k}^n = (D_{\Delta t} H_z)_{i+1/2,j+1/2,k}^n, \quad (52)$$

We define the one-dimensional (1D) discrete scalar product

$$(U, V)_h = \sum_{\alpha} U_{\alpha} V_{\alpha}, \quad \forall (U, V) \in (l^2(\alpha))^2,$$

where α is either an integer or a half-integer. The 3D discrete scalar product will be denoted $((((U, V)))_h$ (or when needed with an index $l^2(\alpha) \times l^2(\beta) \times l^2(\gamma)$).

Discrete integrations by parts yields

$$\begin{cases} (D_{\Delta x} U, V)_{l^2(\alpha)} = -(U, D_{\Delta x} V)_{l^2(\alpha+1/2)}, \forall U \in l^2(\alpha+1/2), V \in l^2(\alpha), \\ (D_{\Delta y} U, V)_{l^2(\beta)} = -(U, D_{\Delta y} V)_{l^2(\beta+1/2)}, \forall U \in l^2(\beta+1/2), V \in l^2(\beta), \\ (D_{\Delta z} U, V)_{l^2(\gamma)} = -(U, D_{\Delta z} V)_{l^2(\gamma+1/2)}, \forall U \in l^2(\gamma+1/2), V \in l^2(\gamma), \end{cases} \quad (53)$$

$$\begin{aligned} & (((((D_{\Delta t} U)^{n+1/2}, \frac{U^n + U^{n+1}}{2})))_h \\ &= \frac{1}{2\Delta t} (\|U^{n+1}\|_h^2 - \|U^n\|_h^2), \quad \forall U \in l^2(\alpha) \times l^2(\beta) \times l^2(\gamma). \end{aligned} \quad (54)$$

Since σ is constant,

$$D_{\Delta t}^\sigma D_{\Delta t} = D_{\Delta t} D_{\Delta t}^\sigma, \quad (55)$$

$$D_{\Delta t}^\sigma D_{\Delta x} = D_{\Delta x} D_{\Delta t}^\sigma, \quad (56)$$

$$D_{\Delta t}^\sigma D_{\Delta y} = D_{\Delta y} D_{\Delta t}^\sigma, \quad (57)$$

$$D_{\Delta t}^\sigma D_{\Delta z} = D_{\Delta z} D_{\Delta t}^\sigma. \quad (58)$$

Theorem 1 For any integer $n \geq 0$, let $E_{i,j,k}^n = (E_{x_{i,j,k}}^n, E_{y_{i,j,k}}^n, E_{z_{i,j,k}}^n)$ and $H_{i,j,k}^n = (H_{x_{i,j,k}}^n, H_{y_{i,j,k}}^n, H_{z_{i,j,k}}^n)$ be

the solution of (45)-(52), then the discrete version of the energy evolution law is,

$$\varepsilon_1^{n+1/2} - \varepsilon_1^{n-1/2} + 2\sigma \|\overline{D_{\Delta t} E_x^n}\|_h^2 + 2\sigma \|\overline{D_{\Delta t} E_y^n}\|_h^2 + \sigma((\overline{D_{\Delta x} E_y^n} - \overline{D_{\Delta y} E_x^n}, D_{\Delta t}^{\sigma} H_z^n))_h = 0, \quad (59)$$

where

$$\begin{aligned} \varepsilon_1^{n+1/2} = & \frac{1}{2\Delta t} \left(\|(D_{\Delta t} E_x)^{n+1/2}\|_h^2 + \|(D_{\Delta t} E_y)^{n+1/2}\|_h^2 + \|(D_{\Delta t} E_z)^{n+1/2}\|_h^2 \right. \\ & \left. + \|\sigma \frac{E_x^n + E_x^{n+1}}{2}\|_h^2 + \|\sigma \frac{E_y^n + E_y^{n+1}}{2}\|_h^2 \right. \\ & \left. + (((D_{\Delta t}^{\sigma} H_x)^{n+1}, (D_{\Delta t}^{\sigma} H_x)^n))_h + (((D_{\Delta t}^{\sigma} H_y)^{n+1}, (D_{\Delta t}^{\sigma} H_y)^n))_h \right). \end{aligned}$$

Proof 1 We divide it into seven parts

(i) By applying $D_{\Delta t}^{\sigma}$ to (45), we get

$$((D_{\Delta t}^{\sigma})^2 E_x)^n + (D_{\Delta z} D_{\Delta t}^{\sigma} H_y)^n - (D_{\Delta y} D_{\Delta t}^{\sigma} H_z)^n = 0. \quad (60)$$

We multiply (60) with $\overline{(D_{\Delta t} E_x)^n}$ to get

$$((((D_{\Delta t}^{\sigma})^2 E_x)^n + (D_{\Delta z} D_{\Delta t}^{\sigma} H_y)^n - (D_{\Delta y} D_{\Delta t}^{\sigma} H_z)^n, \overline{(D_{\Delta t} E_x)^n}))_h = 0. \quad (61)$$

(ii) By applying $D_{\Delta t}^{\sigma}$ to (46). Since σ is constant, we get

$$((D_{\Delta t}^{\sigma})^2 E_y)^n + (D_{\Delta x} D_{\Delta t}^{\sigma} H_z)^n - (D_{\Delta z} D_{\Delta t}^{\sigma} H_x)^n = 0. \quad (62)$$

Multiplying (62) with $\overline{(D_{\Delta t} E_y)^n}$ yields

$$((((D_{\Delta t}^{\sigma})^2 E_y)^n + (D_{\Delta x} D_{\Delta t}^{\sigma} H_z)^n - (D_{\Delta z} D_{\Delta t}^{\sigma} H_x)^n, \overline{(D_{\Delta t} E_y)^n}))_h = 0. \quad (63)$$

(iii) After applying $D_{\Delta t}^{\sigma}$ to (47), then the equation (47) be shift to time n

$$D_{\Delta t}^{\sigma} D_{\Delta t} E_z^* + D_{\Delta t}^{\sigma} D_{\Delta y} H_x - D_{\Delta t}^{\sigma} D_{\Delta x} H_y = 0.$$

Using (57), (58) and (48), this is equivalent to

$$(D_{\Delta t}^2 E_z)^n + (D_{\Delta y} D_{\Delta t}^{\sigma} H_x)^n - (D_{\Delta x} D_{\Delta t}^{\sigma} H_y)^n = 0. \quad (64)$$

We multiply (64) with $\overline{(D_{\Delta t} E_z)^n}$ to get

$$((((D_{\Delta t}^2 E_z)^n + (D_{\Delta y} D_{\Delta t}^{\sigma} H_x)^n - (D_{\Delta x} D_{\Delta t}^{\sigma} H_y)^n, \overline{(D_{\Delta t} E_z)^n}))_h = 0. \quad (65)$$

(iv) Equation (49) is written as (45), (46) and (47) not at time $n+1/2$ but at time n . It is thus necessary to consider the mean-value of (49) at n and $n+1$:

$$\overline{(D_{\Delta t}^{\sigma} H_x)^{n+1/2}} + \overline{(D_{\Delta y} E_z)^{n+1/2}} - \overline{(D_{\Delta z} E_y)^{n+1/2}} = 0.$$

We apply $D_{\Delta t}$ to this equation and multiply it by $(D_{\Delta t}^{\sigma} H_x)^n$ to get

$$\begin{aligned} & (((\overline{(D_{\Delta t}^{\sigma} H_x)^{n+1/2}} + \overline{(D_{\Delta y} D_{\Delta t} E_z)^{n+1/2}} - \overline{(D_{\Delta z} D_{\Delta t} E_y)^{n+1/2}}, (D_{\Delta t}^{\sigma} H_x)^n))_h = 0. \end{aligned} \quad (66)$$

(v) In the same way, we consider the mean-value of (50) at n and $n+1$:

$$\overline{(D_{\Delta t}^{\sigma} H_y)^{n+1/2}} + \overline{(D_{\Delta z} E_x)^{n+1/2}} - \overline{(D_{\Delta x} E_z)^{n+1/2}} = 0.$$

We apply $D_{\Delta t}$ to this equation and multiply it by $(D_{\Delta t}^{\sigma} H_y)^n$ to get

$$\begin{aligned} & (((\overline{(D_{\Delta t}^{\sigma} H_y)^{n+1/2}} + \overline{(D_{\Delta z} D_{\Delta t} E_x)^{n+1/2}} - \overline{(D_{\Delta x} D_{\Delta t} E_z)^{n+1/2}}, (D_{\Delta t}^{\sigma} H_y)^n))_h = 0. \end{aligned} \quad (67)$$

(vi) Let us consider the mean-value of (51) at n and $n+1$:

$$\overline{(D_{\Delta t}^{\sigma} H_z^*)^{n+1/2}} + \overline{(D_{\Delta x} E_y)^{n+1/2}} - \overline{(D_{\Delta y} E_x)^{n+1/2}} = 0.$$

Using (56), (57) and (52), this is equivalent to

$$\overline{(D_{\Delta t}^2 H_z)^{n+1/2}} + \overline{(D_{\Delta x} E_y)^{n+1/2}} - \overline{(D_{\Delta y} E_x)^{n+1/2}} = 0. \quad (68)$$

Then, we multiply (68) with $D_{\Delta t}^{\sigma} H_z^n$ to get

$$((((\overline{(D_{\Delta t}^2 H_z)^{n+1/2}} + \overline{(D_{\Delta x} E_y)^{n+1/2}} - \overline{(D_{\Delta y} E_x)^{n+1/2}}, D_{\Delta t}^{\sigma} H_z^n))_h. \quad (69)$$

(vii) Adding (61), (63), (65), (66), (67) and (69). Due to (53) we obtain that

$$\begin{aligned} & (((((D_{\Delta t}^{\sigma})^2 E_x)^n, \overline{(D_{\Delta t} E_x)^n}))_h + (((((D_{\Delta t}^{\sigma})^2 E_y)^n, \overline{(D_{\Delta t} E_y)^n}))_h \\ & + (((((D_{\Delta t} D_{\Delta t}^{\sigma} H_x)^n, (D_{\Delta t}^{\sigma} H_x)^n))_h + (((((D_{\Delta t} D_{\Delta t}^{\sigma} H_y)^n, (D_{\Delta t}^{\sigma} H_y)^n))_h \\ & + (((((D_{\Delta t}^2 E_z)^n, \overline{(D_{\Delta t} E_z)^n}))_h + \sigma(((\overline{D_{\Delta x} E_y^n} - \overline{D_{\Delta y} E_x^n}, D_{\Delta t}^{\sigma} H_z^n)))_h = 0. \end{aligned} \quad (70)$$

Thus, we put it into six parts, just as follows:

$$\begin{cases} S^1 + S^2 + S^3 + S^4 + S^5 + S^6 = 0, \\ S^1 = (((D_{\Delta t}^2 E_z)^n, \overline{(D_{\Delta t} E_z)^n}))_h, \\ S^2 = (((((D_{\Delta t}^{\sigma})^2 E_x)^n, \overline{(D_{\Delta t} E_x)^n}))_h, \\ S^3 = (((((D_{\Delta t}^{\sigma})^2 E_y)^n, \overline{(D_{\Delta t} E_y)^n}))_h, \\ S^4 = (((\overline{(D_{\Delta t} D_{\Delta t}^{\sigma} H_x)^n}, (D_{\Delta t}^{\sigma} H_x)^n))_h, \\ S^5 = (((\overline{(D_{\Delta t} D_{\Delta t}^{\sigma} H_y)^n}, (D_{\Delta t}^{\sigma} H_y)^n))_h, \\ S^6 = \sigma(((\overline{D_{\Delta x} E_y^n} - \overline{D_{\Delta y} E_x^n}, D_{\Delta t}^{\sigma} H_z^n)))_h. \end{cases} \quad (71)$$

We now calculate S^j ($j = 1, 2, 3, 4, 5$) explicitly. From (54), it is straightforward to get

$$S^1 = \frac{1}{2\Delta t} \left(\|(D_{\Delta t} E_z)^{n+1/2}\|_h^2 - \|(D_{\Delta t} E_z)^{n-1/2}\|_h^2 \right). \quad (72)$$

There is also no difficulty to rewrite the fourth and fifth terms as

$$S^4 = \frac{1}{2\Delta t} (((((D_{\Delta t}^{\sigma} H_x)^{n+1}, (D_{\Delta t}^{\sigma} H_x)^n))_h - (((((D_{\Delta t}^{\sigma} H_x)^n, (D_{\Delta t}^{\sigma} H_x)^{n-1}))_h),$$

$$S^5 = \frac{1}{2\Delta t} (((((D_{\Delta t}^{\sigma} H_y)^{n+1}, (D_{\Delta t}^{\sigma} H_y)^n))_h - (((((D_{\Delta t}^{\sigma} H_y)^n, (D_{\Delta t}^{\sigma} H_y)^{n-1}))_h).$$

Concerning the second term, we first develop

$$\begin{aligned} ((D_{\Delta t}^\sigma)^2 E_x)^n &= (D_{\Delta t}^\sigma D_{\Delta t} E_x)^n + \sigma \overline{(D_{\Delta t}^\sigma E_x)^n} \\ &= (D_{\Delta t}^2 E_x)^n + 2\sigma \overline{(D_{\Delta t} E_x)^n} + \sigma^2 \frac{\overline{E_x^{n+1/2}} + \overline{E_x^{n-1/2}}}{2}. \end{aligned}$$

We multiply this expression with $\overline{(D_{\Delta t} E_x)^n}$, and rearrange the last term

$$\begin{aligned} \sigma^2 \left(\left(\frac{\overline{E_x^{n+1/2}} + \overline{E_x^{n-1/2}}}{2}, \overline{(D_{\Delta t} E_x)^n} \right)_h \right) \\ = \frac{\sigma^2}{2\Delta t} (\| \overline{E_x^{n+1/2}} \|_h^2 - \| \overline{E_x^{n-1/2}} \|_h^2). \end{aligned}$$

and get

$$\begin{aligned} S^2 &= \frac{1}{2\Delta t} (\| (D_{\Delta t} E_x)^{n+1/2} \|_h^2 - \| (D_{\Delta t} E_x)^{n-1/2} \|_h^2 \\ &\quad + \sigma^2 \| \overline{E_x^{n+1/2}} \|_h^2 - \sigma^2 \| \overline{E_x^{n-1/2}} \|_h^2) + 2\sigma \overline{\| (D_{\Delta t} E_x)^n \|_h^2}. \end{aligned} \quad (73)$$

The same as S^3 , we can get

$$\begin{aligned} S^3 &= \frac{1}{2\Delta t} (\| (D_{\Delta t} E_y)^{n+1/2} \|_h^2 - \| (D_{\Delta t} E_y)^{n-1/2} \|_h^2 \\ &\quad + \sigma^2 \| \overline{E_y^{n+1/2}} \|_h^2 - \sigma^2 \| \overline{E_y^{n-1/2}} \|_h^2) + 2\sigma \overline{\| (D_{\Delta t} E_y)^n \|_h^2}. \end{aligned} \quad (74)$$

Due to $S^1 + S^2 + S^3 + S^4 + S^5 + S^6 = 0$, then we can obtain the energy evolution law (59). The proof is finished.

2) General multi-symplectic Runge-Kutta methods.

The Runge-Kutta methods for Maxwell equations (5) are

$$Z_{i,j,k,n} = z_{i,j,k}^0 + \Delta t \sum_{s=1}^r a_{n,s} \partial_t Z_{i,j,k,s}, \quad (75)$$

$$z_{i,j,k}^1 = z_{i,j,k}^0 + \Delta t \sum_{s=1}^r b_s \partial_t Z_{i,j,k,s}, \quad (76)$$

$$Z_{i,j,k,n} = z_{0,j,k}^n + \Delta x \sum_{u=1}^{r_1} \bar{a}_{i,u} \partial_x Z_{u,j,k,n}, \quad (77)$$

$$z_{1,j,k}^n = z_{0,j,k}^n + \Delta x \sum_{u=1}^{r_1} \bar{b}_u \partial_x Z_{u,j,k,n}, \quad (78)$$

$$Z_{i,j,k,n} = z_{i,0,k}^n + \Delta y \sum_{v=1}^{r_2} \tilde{a}_{j,v} \partial_y Z_{i,v,k,n}, \quad (79)$$

$$z_{i,1,k}^n = z_{i,0,k}^n + \Delta y \sum_{v=1}^{r_2} \tilde{b}_v \partial_y Z_{i,v,k,n}, \quad (80)$$

$$Z_{i,j,k,n} = z_{i,j,0}^n + \Delta z \sum_{w=1}^{r_3} \hat{a}_{k,w} \partial_z Z_{i,j,w,n}, \quad (81)$$

$$z_{i,j,1}^n = z_{i,j,0}^n + \Delta z \sum_{w=1}^{r_3} \hat{b}_w \partial_z Z_{i,j,w,n}. \quad (82)$$

Theorem 2 (Hong et al. 2005) If in the methods (75)-(82)

$$b_s b_n - b_s a_{s,n} - b_n a_{n,s} = 0, \quad (83)$$

$$\bar{b}_u \bar{b}_i - \bar{b}_u \bar{a}_{u,i} - \bar{b}_i \bar{a}_{i,u} = 0, \quad (84)$$

$$\tilde{b}_v \tilde{b}_j - \tilde{b}_v \tilde{a}_{v,j} - \tilde{b}_j \tilde{a}_{j,v} = 0, \quad (85)$$

and

$$\hat{b}_w \hat{b}_k - \hat{b}_w \hat{a}_{w,k} - \hat{b}_k \hat{a}_{k,w} = 0, \quad (86)$$

hold for $s, n = 1, 2, \dots, r$, $u, i = 1, 2, \dots, r_1$, $v, j = 1, 2, \dots, r_2$ and $w, k = 1, 2, \dots, r_3$, then the method (75)-(82) is multi-symplectic.

Let $U = (E_{xy}, E_{xz}, E_{yz}, E_{yx}, E_{xy}, E_{xz}, H_{xy}, H_{xz}, H_{yz}, H_{yx}, H_{xy}, H_{xz})^T$, then the 3D Maxwell equations with PML (21)-(32) can be rewritten as

$$\partial_t U + \Sigma U = P(\nabla)U, \quad (87)$$

with

$$\Sigma = \begin{pmatrix} A_{6 \times 6} & 0 \\ 0 & A_{6 \times 6} \end{pmatrix}, \quad P(\nabla) = \begin{pmatrix} 0 & B_{6 \times 6} \\ -B_{6 \times 6} & 0 \end{pmatrix},$$

where

$$A_{6 \times 6} = \begin{pmatrix} 0 & & & & & \\ & \sigma & & & & \\ & & \sigma & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix},$$

$$B_{6 \times 6} = \begin{pmatrix} & & & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} & \\ & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial z} & \\ & & & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \\ -\frac{\partial}{\partial y} & -\frac{\partial}{\partial y} & & & & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial x} \end{pmatrix}.$$

Based on theorem (2), we apply an s -stage symplectic Runge-Kutta method to the t -direction, of form (87), to obtain the following scheme,

$$U^n = u^0 + \Delta t \sum_{m=1}^s a_{nm} (-\Sigma U^m + P(\nabla)U^m), \quad (88)$$

$$u^1 = u^0 + \Delta t \sum_{m=1}^s b_m (-\Sigma U^m + P(\nabla)U^m), \quad (89)$$

where the coefficients of the equations (88)-(89) satisfy the following symplectic condition:

$$b_m a_{mn} + b_n a_{nm} = b_m b_n, \quad m, n = 1, 2, \dots, s. \quad (90)$$

Now, we introduce the following difference operators and apply Yee method to the x -direction, y -direction and z -direction respectively, of form (88)-(89).

$$\delta_x V_{i,j,k}^n = \frac{V_{i+1/2,j,k}^n - V_{i-1/2,j,k}^n}{\Delta x},$$

$$\delta_y V_{i,j,k}^n = \frac{V_{i,j+1/2,k}^n - V_{i,j-1/2,k}^n}{\Delta y},$$

$$\delta_z V_{i,j,k}^n = \frac{V_{i,j,k+1/2}^n - V_{i,j,k-1/2}^n}{\Delta z}.$$

Then, we obtain that,

$$U_{i,j,k}^n = u_{i,j,k}^0 + \Delta t \sum_{m=1}^s a_{nm} \left(-\Sigma U_{i,j,k}^m + \widetilde{P(\nabla)} U_{i,j,k}^m \right), \quad (91)$$

$$u_{i,j,k}^1 = u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m \left(-\Sigma U_{i,j,k}^m + \widetilde{P(\nabla)} U_{i,j,k}^m \right), \quad (92)$$

with

$$\widetilde{P(\nabla)} = \begin{pmatrix} & & \delta y & \delta y \\ & -\delta z & -\delta z & \\ \delta z & \delta z & & \\ & \delta x & \delta x & -\delta x & -\delta x \\ -\delta y & -\delta y & & \end{pmatrix}.$$

Theorem 3 For and integer $n \geq 0$, set $\mathbf{E}_{i,j,k}^n = (E_{x_{i,j,k}}^n, E_{y_{i,j,k}}^n, E_{z_{i,j,k}}^n)$ and $\mathbf{H}_{i,j,k}^n = (H_{x_{i,j,k}}^n, H_{y_{i,j,k}}^n, H_{z_{i,j,k}}^n)$ be the solution of the discrete scheme (91)-(92), then the discrete version of the energy evolution law is,

$$\|\mathbf{E}^{n+1}\|_h^2 + \|\mathbf{H}^{n+1}\|_h^2 = \|\mathbf{E}^n\|_h^2 + \|\mathbf{H}^n\|_h^2 - 2\Delta t \sigma \sum_{m=1}^s b_m ((E_{i,j,k}^m)^2 + (H_{i,j,k}^m)^2), \quad (93)$$

where

$$\begin{aligned} \|\mathbf{E}^n\|_h^2 &= h_x h_y h_z \sum_{i=r_1}^s \sum_{j=r_2}^s \sum_{k=r_3}^s ((E_{x_{i,j,k}}^n)^2 + (E_{y_{i,j,k}}^n)^2 + (E_{z_{i,j,k}}^n)^2), \\ \|\mathbf{H}^n\|_h^2 &= h_x h_y h_z \sum_{i=r_1}^s \sum_{j=r_2}^s \sum_{k=r_3}^s ((H_{x_{i,j,k}}^n)^2 + (H_{y_{i,j,k}}^n)^2 + (H_{z_{i,j,k}}^n)^2). \end{aligned}$$

Proof 2 For simplicity, we introduce the following notation

$$f_{i,j,k}^m = -\Sigma U_{i,j,k}^m + \widetilde{P(\nabla)} U_{i,j,k}^m. \quad (94)$$

From the discrete scheme (92), we can get that the following relation between $u_{i,j,k}^1$ and $u_{i,j,k}^0$:

$$\begin{aligned} (u_{i,j,k}^1)^T u_{i,j,k}^1 &= (u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m f_{i,j,k}^m)^T (u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m f_{i,j,k}^m) \\ &= ((u_{i,j,k}^0)^T u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m (f_{i,j,k}^m)^T (u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m f_{i,j,k}^m) \\ &= (u_{i,j,k}^0)^T u_{i,j,k}^0 + \Delta t ((u_{i,j,k}^0)^T \sum_{m=1}^s b_m f_{i,j,k}^m) \\ &\quad + \Delta t (\sum_{m=1}^s b_m (f_{i,j,k}^m)^T u_{i,j,k}^0) + (\Delta t)^2 \sum_{m,n=1}^s b_m b_n (f_{i,j,k}^m)^T f_{i,j,k}^n. \end{aligned}$$

Note that the discrete scheme (91) and the notation (94), it can be rewritten as

$$u_{i,j,k}^0 = U_{i,j,k}^n - \Delta t \sum_{m=1}^s a_{nm} f_{i,j,k}^m. \quad (95)$$

Then, we insert the relation between $u_{i,j,k}^1$ and $u_{i,j,k}^0$ into (95), we obtain that

$$\begin{aligned} &(u_{i,j,k}^1)^T u_{i,j,k}^1 \\ &= (u_{i,j,k}^0)^T u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m ((U_{i,j,k}^n)^T f_{i,j,k}^m + (f_{i,j,k}^m)^T U_{i,j,k}^n) \\ &\quad + (\Delta t)^2 \sum_{m,n=1}^s (b_m b_n - b_m a_{nm} - b_n a_{nm}) (f_{i,j,k}^m)^T f_{i,j,k}^n. \end{aligned} \quad (96)$$

Due to the symplectic condition (90), we have

$$\begin{aligned} (u_{i,j,k}^1)^T u_{i,j,k}^1 &= (u_{i,j,k}^0)^T u_{i,j,k}^0 + \Delta t \sum_{m=1}^s b_m ((U_{i,j,k}^n)^T f_{i,j,k}^m \\ &\quad + (f_{i,j,k}^m)^T U_{i,j,k}^n). \end{aligned} \quad (97)$$

Since the Maxwell equations energy conserving law in lossless medium, we obtain

$$(U_{i,j,k}^m)^T (\widetilde{P(\nabla)} U_{i,j,k}^m) = 0.$$

Therefore, (97) becomes the following form

$$(u_{i,j,k}^1)^T u_{i,j,k}^1 = (u_{i,j,k}^0)^T u_{i,j,k}^0 - 2\Delta t \sigma \sum_{m=1}^s (b_m (U_{i,j,k}^m)^T \Sigma U_{i,j,k}^m). \quad (98)$$

From the presentation of Σ and U , we have

$$(u_{i,j,k}^1)^T u_{i,j,k}^1 = (u_{i,j,k}^0)^T u_{i,j,k}^0 - 2\Delta t \sigma \sum_{m=1}^s b_m ((E_{i,j,k}^m)^2 + (H_{i,j,k}^m)^2). \quad (99)$$

Summing all terms in the above equation (99) over all spatial indices i, j, k , then we can get the energy evolution law (93). The proof is finished.

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